SOLUTION OF A DIRICHLET PROBLEM IN A CRESCENT-SHAPED DOMAIN

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We present a solution of a Dirichlet problem for the Laplace equation in a crescent-shaped domain and apply this solution to some stationary problems of heat conduction, electrostatics, and the theory of elasticity.

1. Let g (Fig. 1) be a crescent-shaped domain in the complex w plane:

$$g=\{w:|w|<R;\ |w-O_1|>\eta\}, \tag{1}$$

$$O_1=-R\frac{\sin\alpha\pi}{\sin\nu\pi},\quad \eta=R\frac{\sin\frac{\beta\pi}{2}}{\sin\nu\pi},\quad \nu=\frac{\beta}{2}-\alpha.$$

It is uniquely defined by the parameters $R\in(0,\infty)$, $\beta\in(1,2)$, $\alpha\in\left(0,\frac{\beta}{2}\right)$. Its boundary ∂g consists of two components, namely, the circular arcs γ and Γ:

$$\gamma=\{w:w=O_1+\eta e^{i\varphi_1},\ \varphi_1\in[-\nu\pi,\ \nu\pi]\}, \tag{2}$$

$$\Gamma=\left\{w:w=Re^{i\varphi},\ \varphi\in\left[-\frac{\beta\pi}{2},\ \frac{\beta\pi}{2}\right]\right\}, \tag{3}$$

which are joined at the points B and C:

$$B=R\exp\left(i\frac{\beta\pi}{2}\right),\quad C=R\exp\left(-i\frac{\beta\pi}{2}\right). \tag{4}$$

To stress the dependence of the domain g on its defining parameters, we write $g(R,\beta,\alpha)$.

Let N be the point of intersection of the arc γ with the line $\{w:\ \mathrm{Im}\,w=0\}$:

$$N=R\frac{\cos\frac{\pi}{2}(\nu+2\alpha)}{\cos\frac{\nu\pi}{2}}, \tag{5}$$

$(NR)=\{w\ \mathrm{Im}\,w=0,\ N\le \mathrm{Re}\,w<R\}$ is a half-open interval of the real axis; int γ is the arc γ minus the endpoints B and C; $f_1\circ f_2$ is the superposition of the functions $f_1(f_2)$.

In the domain g we consider the following Dirichlet problem for the Laplace equation:

$$\Delta\psi(w)=0,\quad w\in g, \tag{6}$$

$$\psi(w)=0,\quad w\in \mathrm{int}\,\gamma, \tag{7}$$

$$\psi(w)=h(\varphi),\quad Re^{i\varphi}=w\in\Gamma, \tag{8}$$

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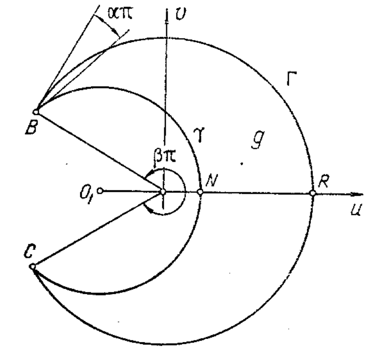

Fig. 1. Crescent-shaped domain.

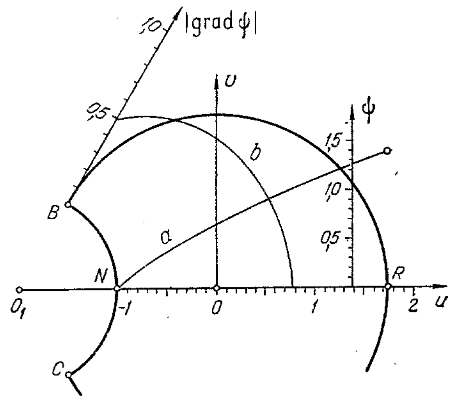

Fig. 2. Solution of the Dirichlet problem and its gradient.

where $h(\varphi)\in L_2\left(-\frac{\beta\pi}{2}, \frac{\beta\pi}{2}\right)$. We are required to find a function $\psi(w)$ in the domain g and the quantity grad $\psi(w)$ on $g \cup \text{int}\,\gamma$. Existence and uniqueness of a solution of the problem (6)-(8) follows from [1].

A solution of this problem may be found (see Sec. 3) in the following way: with the help of a conformal mapping of g onto a semicircle (obtained in Sec. 2), we convert this problem into a Dirichlet problem in a semicircle, which we then solve by the method of separation of variables; we then return, with the help of the inverse mapping, to the domain g to obtain $\psi(w)$. Numerical results are presented in Sec. 4 for the quantities $\psi(w)$, $w \in (NR)$, and $|\text{grad}\ \psi(w)|$, $w \in \gamma$. In Sec. 5 we indicate some generalizations of our solution, and in Sec. 6 we show its relation to the problem of the torsion of a prismatic rod.

2. We denote by $z = \mathcal{F}(w)$ the conformal mapping of the domain g onto the semicircle $\mathcal{F}(g) = \{z: |z|<1,\ 0<\text{Im}\,z\}$, satisfying the conditions

$$\mathcal{F}(B) = -1,\ \mathcal{F}(N) = 0,\ \mathcal{F}(C) = 1, \tag{9}$$

where the points B, N, C are defined by expressions (4) and (5).

To obtain this mapping, we first map the domain g with the aid of the function

$$\zeta(w) = -e^{-i\pi\nu}\,\frac{w-B}{w-C} \tag{10}$$

onto the angular domain $\{\zeta: 0 < \arg\zeta < \alpha\pi\}$, and then, applying a combination of power and fractional-linear functions

$$z(\zeta) = \frac{\zeta^{1/2\alpha} - 1}{\zeta^{1/2\alpha} + 1}, \tag{11}$$

we map the latter onto the semicircle $\mathcal{F}(g)$. The mapping desired is then

$$\mathcal{F}(w) = z(\zeta) \circ \zeta(w). \tag{12}$$

It possesses the property of symmetry, $\mathcal{F}(\bar{w}) = -\overline{\mathcal{F}}(w)$, and maps Γ into the semicircular arc

$$\mathcal{F}(\Gamma) = \{z = e^{i\theta} : \theta \in [0, \pi]\}, \tag{13}$$

and maps γ into a diameter of the semicircle

$$\mathcal{F}(\gamma) = \{z = x + iy : x \in [-1, 1], \ y = 0\}, \tag{14}$$

finally, it maps the half-open interval (NR) of the real u axis into the half-open interval of the imaginary y axis:

$$\mathcal{F}((NR)) = \{z = x + iy : x = 0, \ y \in [0, 1)\}. \tag{15}$$

The correspondence between the angular coordinate φ_1, defining a point of the arc γ through Eq. (2), and the abscissa of its image point $\mathcal{F}(\gamma)$ through Eq. (14) is effected by means of the function

$$x(\varphi_1) = \frac{\sin^{1/2\alpha} \frac{1}{2}(\pi\nu - \varphi_1) - \sin^{1/2\alpha} \frac{1}{2}(\pi\nu + \varphi_1)}{\sin^{1/2\alpha} \frac{1}{2}(\pi\nu - \varphi_1) + \sin^{1/2\alpha} \frac{1}{2}(\pi\nu + \varphi_1)}, \tag{16}$$

while the correspondence between the abscissa u of a point of the half-open interval (NR) and the ordinate y of its image (15) is given by the function

$$y(u) = \operatorname{tg}\left\{\frac{1}{2\alpha} \operatorname{arctg} \frac{u\cos\frac{\pi\nu}{2} - R\cos\frac{\pi}{2}(\nu + 2\alpha)}{u\sin\frac{\pi\nu}{2} + R\sin\frac{\pi}{2}(\nu + 2\alpha)}\right\}. \tag{17}$$

The formulas (16) and (17) follow from the relations (10), (11), and (12). The derivative of the mapping function is given by

$$\mathcal{F}'(w) = \frac{R}{\alpha} \sin\frac{\pi\beta}{2}\left\{\frac{w - B}{w - C}\left[e^{i\frac{\pi(1-\nu)}{\alpha}}\left(\frac{w - B}{w - C}\right)^{\frac{1}{\alpha}} + e^{i\frac{\pi(\nu-1)}{\alpha}}\left(\frac{w - C}{w - B}\right)^{\frac{1}{\alpha}}\right]^2\right\}^{-1}. \tag{18}$$

We obtain the inverse mapping $w = \mathcal{F}^{-1}(z)$, starting from the relations (10), (11), and (12), in the form

$$\mathcal{F}^{-1}(z) = w(\zeta) \circ \zeta(z), \tag{19}$$

where

$$w(\zeta) = C\frac{\zeta + e^{i\pi(\nu + 2\alpha)}}{\zeta + e^{-i\pi\nu}}, \tag{20}$$

$$\zeta(z) = \left(\frac{1 + z}{1 - z}\right)^{2\alpha}. \tag{21}$$

From this we obtain the modulus of its derivative on the intervals $\mathcal{F}(\gamma)$ (see Eq. (14)) and $\mathcal{F}((NR))$ (see Eq. (15)):

$$|\mathcal{F}^{-1\prime}(x)| = 8\alpha R \sin\frac{\pi\beta}{2}(1-x^2)^{2\alpha-1} \times$$
$$\times [(1+x)^{4\alpha} + 2\cos\pi\nu(1-x^2)^{2\alpha} + (1-x)^{4\alpha}]^{-1},\ x \in \mathcal{F}(\gamma); \tag{22}$$

$$|\mathcal{F}^{-1\prime}(iy)| = 2\alpha R \sin\frac{\pi\beta}{2}\left[(1+y^2)\cos^2\left(\frac{\pi\nu}{2} + 2\alpha\,\mathrm{arctg}\,y\right)\right]^{-1}. \tag{23}$$

The correspondence between the angular coordinate θ, defining a point of the arc $\mathcal{F}(\Gamma)$ (see Eq. (13)), and the angular coordinate φ of its preimage on Γ (see Eq. (3)) has the form

$$\varphi(\theta) = 2\,\mathrm{arctg}\left\{\mathrm{tg}\frac{\pi\beta}{4}\,\frac{\sin^{2\alpha}\frac{\theta}{2} - \cos^{2\alpha}\frac{\theta}{2}}{\sin^{2\alpha}\frac{\theta}{2} + \cos^{2\alpha}\frac{\theta}{2}}\right\}. \tag{24}$$

We establish this as follows. Putting $z = e^{i\theta}$ into Eq. (21) in accordance with Eq. (13), we find

$$\zeta(z) = e^{i\alpha\pi}\,\mathrm{ctg}^{2\alpha}\frac{\theta}{2},\quad e^{i\theta} = z \in \mathcal{F}(\Gamma); \tag{25}$$

substituting $w = Re^{i\varphi}$ into Eq. (20) in accordance with Eq. (3), and also the relation (25), we obtain

$$e^{i\varphi} = \frac{e^{-i\frac{\pi\beta}{2}}\,\mathrm{ctg}^{2\alpha}\frac{\theta}{2} + 1}{\mathrm{ctg}^{2\alpha}\frac{\theta}{2} + e^{-i\frac{\pi\beta}{2}}};$$

multiplying numerator and denominator by the conjugate of the denominator and calculating the argument of both sides of the equation, we obtain the relation (24).

3. It is not difficult to see that the function $\psi_z(z) = \psi(\mathcal{F}^{-1}(z))$ satisfies the conditions of the problem:

$$\Delta\psi_z(z) = 0,\quad z \in \mathcal{F}(g), \tag{26}$$
$$\psi_z(z) = 0,\quad z \in \mathcal{F}(\mathrm{int}\,\gamma), \tag{27}$$
$$\psi_z(z) = h(\varphi(\theta)),\quad e^{i\theta} = z \in \mathcal{F}(\Gamma), \tag{28}$$

where φ(θ) is given by the expression (24), and where $\mathcal{F}(\gamma)$ and $\mathcal{F}(\Gamma)$ are, respectively, the semicircle and diameter of the semicircle $\mathcal{F}(g)$. We obtain the solution of the problem (26)-(28) with the aid of the method of separation of variables:

$$\psi_z(z) = \sum_{p=1}^{\infty} a_p \rho^p \sin p\theta,\quad \rho e^{i\theta} = z \in \mathcal{F}(g), \tag{29}$$

where the coefficients a_p are calculated from the formula

$$a_p = \frac{2}{\pi}\int_0^{\pi} h(\varphi(\theta))\sin p\theta\, d\theta. \tag{30}$$

Substituting $z = \mathcal{F}(w)$ into Eq. (29) and noting that $\rho^p \sin p\theta = \operatorname{Im} z^p$, we obtain the solution of the problem (6)-(8) in the form of the series

$$\psi(w) = \sum_{p=1}^{\infty} a_p \operatorname{Im}[\mathcal{F}(w)]^p; \tag{31}$$

with the help of [1] we can show that this series converges in $g \cup \operatorname{int}\gamma$, wherein it can be differentiated an arbitrary number of times. We obtain an expression for the gradient of the solution $\psi(w)$ from Eq. (31) through use of the equation grad [Im f(w)] = if'(w), valid for an arbitrary analytic function f(w):

$$\operatorname{grad}\psi(w) = i\mathcal{F}'(w)\sum_{p=1}^{\infty} pa_p[\mathcal{F}(w)]^{p-1}, \quad w \in g \cup \operatorname{int}\gamma. \tag{32}$$

We note that $\operatorname{grad}\psi(N) = ia_1\mathcal{F}'(N)$.

Equation (31) for $w \in (NR)$ and expression (32) for $w \in (NR)$ and $w \in \operatorname{int}\gamma$ can be put into a more suitable form. Taking into consideration, in accordance with Eq. (15), that $\mathcal{F}(u) = iy(u)$, $u \in (NR)$, we transform Eq. (31):

$$\psi(u) = \sum_{p=1}^{\infty} a_{2p-1}(-1)^{p+1}[y(u)]^{2p-1}, \quad u = w \in (NR), \tag{33}$$

where y(u) is given by the function (17). Noting that the normal $\nu(w)$ to the arc γ is determined, in accordance with Eq. (2), by the expression

$$\nu(w) = e^{i\varphi_1}, \quad O_1 + \eta e^{i\varphi_1} = w \in \gamma,$$

and taking into account the relation $\mathcal{F}'(w) = \frac{i}{\nu(w)}|\mathcal{F}^{-1\prime}(x)|^{-1}$, $\mathcal{F}^{-1}(x) = w \in \gamma$, we deduce from Eq. (32) the expression

$$\operatorname{grad}\psi(w) = \frac{\nu(w)}{|\mathcal{F}^{-1\prime}(x(\varphi_1))|}\sum_{p=1}^{\infty} pa_p[x(\varphi_1)]^{p-1}, \quad \operatorname{int}\gamma, \tag{34}$$

where $|\mathcal{F}^{-1\prime}(x)|$ is determined from Eq. (22) and $x(\varphi_1)$ is determined from Eq. (16). Substituting $\mathcal{F}(u) = iy(u)$, $u \in (NR)$ into Eq. (32) and using the equation $\mathcal{F}'(u) = i|\mathcal{F}^{-1\prime}(iy(u))|^{-1}$, $u \in (NR)$, we obtain

$$\operatorname{grad}\psi(u) = \frac{1}{|\mathcal{F}^{-1\prime}(iy(u))|}\Big\{\sum_{p=1}^{\infty}(-1)^{p-1}(2p-1)a_{2p-1}[y(u)]^{2(p-1)} + i\sum_{p=1}^{\infty}(-1)^p 2pa_{2p}[y(u)]^{2p-1}\Big\}, \quad u = w \in (NR), \tag{35}$$

where $|\mathcal{F}^{-1\prime}(iy)|$ is determined from Eq. (23), and y(u) from Eq. (17). With the help of the relations (23), (24), and (35) we obtain the gradient at the point N:

$$\operatorname{grad}\psi(N) = \frac{1+\cos\pi\nu}{2\alpha\pi R\sin\frac{\pi\beta}{2}} \times \int_0^{\pi} h\left(2\operatorname{arctg}\left[\operatorname{tg}\frac{\pi\beta}{4}\,\frac{\sin^{2\alpha}\frac{\theta}{2} - \cos^{2\alpha}\frac{\theta}{2}}{\sin^{2\alpha}\frac{\theta}{2} + \cos^{2\alpha}\frac{\theta}{2}}\right]\right)\sin\theta\, d\theta. \tag{36}$$

4. Numerical calculations make it possible to obtain the solution for various values of the initial parameters R, β, α and for various forms of the function h(φ). Our numerical results are shown in Fig. 2 for the case in which $R = \sqrt{3}$, $\beta = 5/3$, $\alpha = 1/2$, $h(\varphi) = 3^{0.3}\cdot\cos(0.6\,\varphi)$; the heavily drawn curve exhibits (partially) the boundary of the domain g; curve b shows the value of $|\mathrm{grad}\,\psi(w)|$ on the arc γ (its values are laid off along the normal to the curve γ); curve a is the graph of the function ψ(w) on the segment (NR).

5. Restrictions on the parameters β and α, adopted in Section 1, were for the sake of convenience. Our results, with small changes, can be carried over to arbitrary $\beta \in (0, 2)$, $\alpha \in (-\infty, \infty)$; in the main, our results also carry over to the case of an arbitrary nonunivalent circular lune. We note that for $-2 < \alpha < 0$ the domain g(R, β, α) is complementary to the closure of the domain g(R, β, 2 + α) with respect to the extended complex plane.

Let the domain g = g(R, β, α) and the parameters R, β, α satisfy the conditions of Sec. 1, and let the transformation $w = f(\tilde{w})$ and the quantities $\tilde{R}$, $\tilde{\beta}$, $\tilde{\alpha}$ be given by the expressions

$$\tilde{\alpha} = -\alpha,\ \tilde{\beta} = \beta - 2\alpha,\ \tilde{R} = \eta(R, \beta, \alpha),$$

$$f(\tilde{w}) = \tilde{w} - \tilde{O}_1(\tilde{R}, \tilde{\beta}, \tilde{\alpha}), \tag{37}$$

where the function $\tilde{O}_1(\tilde{R}, \tilde{\beta}, \tilde{\alpha})$ is obtained from Eq. (1) by affixing the tilde sign to the corresponding symbols. We then denote the crescent-shaped domain defined by the parameters $\tilde{R}$, $\tilde{\beta}$, $\tilde{\alpha}$ by $\tilde{g}$, and its bounding arcs by $\tilde{\Gamma}$ and $\tilde{\gamma}$. It is not difficult to see that $g = f(\tilde{g})$, $\Gamma = f(\tilde{\gamma})$ and $\gamma = f(\tilde{\Gamma})$; therefore the solution of the problem

$$\Delta\psi_1(w) = 0, \quad w \in g, \tag{38}$$

$$\psi_1(w) = 0, \quad w \in \mathrm{int}\,\Gamma, \tag{39}$$

$$\psi_1(w) = h_1(w), \quad w \in \gamma, \quad h_1 \in L_2(\gamma), \tag{40}$$

is reduced with the aid of the transformation (37) to the problem (6)-(8) considered above, with g, ψ(w), h(φ) replaced, respectively, by $\tilde{g}$, $\tilde{\psi}(\tilde{w}) = \psi_1(f(\tilde{w}))$, $\tilde{h}(\varphi) = h_1(f(\tilde{R}e^{i\varphi}))$. Consequently, we can also find a solution of the Dirichlet problem in the crescent-shaped domain g with an arbitrary $L_2(\partial g)$ function on its boundary since it is obviously equal to the sum of the solutions of the problems (6)-(8) and (38)-(40).

6. We can reduce the problem of the torsion of a prismatic rod with cross section in the form of the domain g (see [2-5]) to a particular case of the problem (6)-(8); moreover, the function h(φ) in the condition (8) has the form

$$h(\varphi) = \frac{1}{2}\left(|Re^{i\varphi} - O_1|^2 - \eta^2\right) = R^2\,\frac{\sin\alpha\pi}{\sin\nu\pi}\left(\cos\varphi - \cos\frac{\beta\pi}{2}\right). \tag{41}$$

The maximum stress τ_{max} is reached at the point N; it is connected with grad ψ(N) by means of the relationship

$$\tau_{\max} = G\mu\,[-\eta + \mathrm{grad}\,\psi(N)], \tag{42}$$

where G is the shear modulus of the rod material and μ is the angle of torsion of the rod per unit length. Using the relations (36), (41), (42), we obtain a formula for τ_{max}:

$$\tau_{\max} = G\mu\eta\left\{-1 + \frac{(1+\cos\nu\pi)\sin\alpha\pi}{\alpha\pi\sin^2\frac{\beta\pi}{2}} \times \int_0^{\pi/2}\left[\cos\varphi(\theta) - \cos\frac{\beta\pi}{2}\right]\sin\theta\,d\theta\right\},$$

where φ(θ) is given by expression (24).

The torsion problem for a rod of this type was solved in [2, 3]. Particular cases of this problem were considered in other publications: the case α = 1/2 in [4]; α = β/4 in [5]; α = 1/2, β = 1/2 and α = 1/2, β = 1 in [6]; α = 1/n, where n = 2, 3, ..., in [7].

NOTATION

w, z, ζ, complex variables; i, imaginary unit; $w = u + iv = re^{i\varphi}$, where u, v and r, φ, respectively, are systems of Cartesian and polar coordinates over the plane w; $z = x + iy = \rho e^{i\theta}$, where x, y and ρ, θ, respectively, are systems of Cartesian and polar coordinates over the plane z; grad, gradient; Δ, Laplace operator: $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ or $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; $f_1 \circ f_2$, superposition of functions $f_1(f_2)$; int γ, arc γ without end points; ∂g, boundary of the domain g; $\bar{a}$, number, complex conjugate to a; Re a and Im a, real and imaginary parts of the number a, respectively; $L_2(l)$, space of functions summable with a square at intercept l.

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